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Finite-volume corrections to the leading-order hadronic contribution to $g_\mu - 2$

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We present preliminary results of a 2+1-flavor study of finite-volume effects in the lattice QCD computation of the leading-order hadronic contribution to the muon anomalous magnetic moment. We also present methods for obtaining directly the invariant hadronic polarization function, $\Pi(Q^2)$, and the Adler function at all discrete lattice values of Q^2, including $Q^2 = 0$. Results are obtained with HEX-smeared clover fermions.

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1. Introduction

There still is an unexplained 3.4 σ discrepancy between the experimental measurement [1] of the anomalous magnetic moment of the muon, $a_\mu = \frac{(g-2)_\mu}{2}$, and the theoretical determination based on the standard model [2]. The leading uncertainties in the theoretical computation are associated with two non-perturbative QCD corrections : the hadronic vacuum polarization contribution $a_\mu^{\rm HVP,LO}$ at $\mathscr{O}(\alpha)^2$ and the hadronic light-by-light scattering term at $\mathscr{O}(\alpha^3)$. The former is presently best determined via dispersion relations applied to experimental cross sections for e^+e^- annihilation and τ decays into hadrons [3, 4]. In anticipation of the future Fermilab E989 experiment, whose goal is to divide by 4 the error on the measurement of a_μ [5], several groups are now working on computations of $a_\mu^{HVP,LO}$ using lattice QCD simulations (see e.g. for their latest contributions [6, 7, 8, 9, 10, 11]). This approach will provide a valuable *ab-initio* cross-check for phenomenological determinations.

We present a method for obtaining the scalar polarization function, $\Pi(Q^2)$, and the Adler function at all discrete lattice values of Q^2 including zero, directly from the vector-vector correlation function. We then use this approach and the usual one based on the vacuum polarization tensor, to study finite-volume effects in the lattice computation of $\Pi(Q^2)$ and $a_\mu^{\rm HVP,LO}$. In particular, we present preliminary results of a dedicated study of these effects at a fixed lattice spacing of 0.104 fm and pion mass $M_\pi \sim 292\,\mathrm{MeV}$, for lattices ranging in spatial size L from 2.5 to 8.3 fm.

2. Usual and new ways to obtain $a_\mu^{\rm HVP,LO}$ on the lattice

Based on a formula first derived in [12], it was shown in [13] how $a_\mu^{\rm HVP,LO}$ can be obtained from the polarization tensor computed directly in Euclidean spacetime, using lattice QCD simulations. Schematically one computes the Fourier transform of the expectation value of the product of two electromagnetic quark currents, J_μ, at Euclidean lattice momenta Q:

$$\Pi_{\mu\nu}(Q) = a^4 \sum_x \langle J_\mu(x) J_\nu(0)\rangle e^{iQ\cdot x} \ . \tag{2.1}$$

Neglecting Euclidean $O(4)$ violations, the vacuum polarization tensor, $\Pi_{\mu\nu}(Q)$, can be written in terms of a single invariant function $\Pi(Q^2)$, as:

$$\Pi_{\mu\nu}(Q) = (Q_\mu Q_\nu - \delta_{\mu\nu}Q^2)\Pi(Q^2) \ . \tag{2.2}$$

Note that the decomposition assumes that $\Pi_{\mu\nu}(Q=0) = 0$, which is certainly true in infinite volume, and is required for the photon to remain massless. But it is not necessarily the case in a finite spacetime. Thus, we distinguish two methods. In the first, which we call *"usual without subtraction"*, we define $\Pi(Q^2) \equiv \Pi_{\mu\nu}(Q)/(Q_\mu Q_\nu - \delta_{\mu\nu}Q^2)$. In the *usual method with subtraction*, we take $\Pi(Q^2) \equiv [\Pi_{\mu\nu}(Q) - \Pi_{\mu\nu}(0)]/(Q_\mu Q_\nu - \delta_{\mu\nu}Q^2)$ [14]. Here we consider only the spatial, $\Pi_{ii}(Q)$, $i = 1,2,3$, components of the polarization tensor. The resulting $\Pi(Q^2)$ is then fitted as a function of Q^2 and extrapolated to $Q^2 = 0$ to perform the required additive renormalization, $\hat{\Pi}(Q^2) = \Pi(Q^2) - \Pi(0)$. The same fit is used to integrate the polarization function with a known QED kernel $w_\Pi(Q^2)$, yielding the muon anomalous magnetic moment through:

$$a_\mu^{\rm HVP,LO} = 4\pi^2 \left(\frac{\alpha}{\pi}\right)^2 \sum_{f=u,d,s,\ldots} q_f^2 \int_0^\infty dQ^2 w_\Pi(Q^2)\hat{\Pi}_f(Q^2) \ , \tag{2.3}$$

where q_f is the charge of quark flavor f in units of e. $a_\mu^{\mathrm{HVP,LO}}$ is then studied as a function of simulations parameters and interpolated and/or extrapolated to the physical values of the quark masses and to the continuum and infinite volume limits.

A precise determination of $a_\mu^{\mathrm{HVP,LO}}$ on the lattice is particularly challenging because the kernel w_Π peaks at $Q^2 \sim (m_\mu/2)^2$, which is smaller than the lowest non-zero momenta $(2\pi/L,T)$ available in current simulations on $T\times L^3$ lattices with periodic boundary conditions. Moreover, $\Pi_{\mu\nu}(Q)$ is noisy for small values of Q^2.

Besides the *usual* methods for determining $\Pi(Q^2)$ described above, we consider a third approach, which circumvents the problem that $\Pi(Q^2=0)$ is not directly accessible. Thus, it eliminates the systematic and statistical error associated with the extrapolation to $Q^2=0$ required to renormalize $\Pi(Q^2)$ and to describe this function for the values of Q^2 which contribute most to $a_\mu^{\mathrm{HVP,LO}}$.

This second set of methods considers Fourier derivatives of the polarization tensor: [1]

$$\partial_\rho\partial_\sigma\Pi_{\mu\nu}(Q) = -a^4\sum_x x_\rho x_\sigma \langle J_\mu(x)J_\nu(0)\rangle e^{iQ\cdot x}\ . \tag{2.4}$$

We call it the *-2nd derivative-* method. By appropriately choosing the indices and the four-momentum, one can obtain directly the desired polarization scalar through:

$$\Pi(Q^2) = \partial_\mu\partial_\nu\Pi_{\mu\nu}(Q)|_{Q_\mu=Q_\nu=0}, \qquad \mu\neq\nu\ , \tag{2.5}$$

or through

$$\Pi(Q^2) = -\frac{1}{2}\partial_\mu\partial_\mu\Pi_{\nu\nu}(Q)|_{Q_\rho=0,\rho\neq\nu}, \qquad \mu\neq\nu\ . \tag{2.6}$$

One can also obtain the Adler function:

$$\mathscr{A}(Q^2) = Q^2\frac{\partial\Pi(Q^2)}{\partial Q^2} = -\frac{1}{2}\partial_\mu\partial_\mu\Pi_{\mu\mu}(Q)|_{Q_\mu=0}\ . \tag{2.7}$$

In all cases, one may choose to sum over repeated indices, including spatial and/or temporal components, depending on the symmetries of the lattice under consideration. Here we focus on the polarization scalar and we consider only results obtained from Eq. (2.5) with the spatial components, $\Pi_{ij}(Q)$, $i\neq j=1,2,3$, of the polarization tensor. In general, the *2nd derivative* methods have the advantage that they also work to obtain information at $Q^2=0$, thereby guaranteeing that the interesting values of Q^2 are reached by a controlled interpolation. A possible drawback is that the factor of $x_\rho x_\sigma$ term, in Eq. (2.4), emphasizes long-distance contributions which are more noisy and more subject to finite-volume effects.

In both the *usual* and *2nd derivative* approaches we split up the fit of $\Pi(Q^2)$ vs Q^2 into two regions, as suggested in [11]. In the *-low-* Q^2 region, we fit a [1,1], three parameter Padé to the 4 lowest available momentum points. The *-high-* Q^2 results up to $1\,\mathrm{GeV}^2$ are fit to another Padé. In that way, the fit to the low region, which contributes most to $a_\mu^{\mathrm{HVP,LO}}$, and has larger statistical errors, is not distorted by the more precise results at higher values of Q^2. The integration yielding $a_\mu^{\mathrm{HVP,LO}}$ is split accordingly, with no particular matching at $0.2\,\mathrm{GeV}^2$. Note that, in this proceedings, we only integrate up to $Q^2=1\,\mathrm{GeV}^2$ and thus consider a quantity $a_\mu^{\mathrm{HVP,LO}}(Q^2\leq 1\,\mathrm{GeV}^2)$ which is not quite the HVP contribution to the muon anomalous magnetic moment.

[1] Fourier derivatives are also considered in [9, 10], for instance, but only in the time direction and at $Q^2=0$.

3. Finite-volume study

The preliminary finite-volume study presented here is based on four Budapest-Marseille-Wuppertal, $N_f = 1+1+1+1$ ensembles that were generated for the recent calculation of the neutron-proton mass difference in QCD+QED [15]. Here we consider $N_f = 2+1$ valence flavors that couple only to the $SU(3)$-color components of the links, with bare masses adjusted to reproduce the isospin averaged u and d quark and the s quark masses of the $N_f = 1+1+1+1$ calculation. The results are obtained using a tree-level $\mathscr{O}(a^2)$ improved Symanzik gauge action, together with tree-level clover-improved Wilson fermions. The gluon fields undergo three steps of HEX smearing before being coupled to the quarks. The hadronic vacuum polarization tensor, $\Pi_{\mu\nu}$, is computed with a local vector current at the source (indexed by ν) and a conserved vector current at the sink (indexed by μ). For the present study we neglect disconnected components, which are expected to be small compared to the connected contributions that we retain, and which should not appreciably modify the finite-volume behavior.

The four ensembles considered here are those from [15] with $\beta = 3.2$, corresponding to $a = 0.104\,\mathrm{fm}$, and with $M_\pi \sim 292\,\mathrm{MeV}$. The bare mass parameters used in the valence sector are $am_{ud} = -0.077$ and $am_s = -0.050$. The four ensembles differ only in their volumes, with the spatial size of spacetime, L, ranging from 2.5 to 8.3 fm. The relevant characteristics of the ensembles are:

T/a	L/a	M_π (MeV)	T (fm)	L (fm)	M_πT	M_πL
48	24	295.2(1.4) (0.50%)	5.0	2.5	7.5	3.7
64	32	292.6(7) (0.23%)	6.7	3.3	9.9	4.9
96	48	292.0(6) (0.20%)	10.0	5.0	14.8	7.4
64	80	292.1(3) (0.12%)	6.7	8.3	9.9	12.3

Three of the four simulations have $T/L = 2$ and all are on asymetrical lattices. The pion is light enough to allow the ρ to decay into two pions in the infinite volume limit. It is important to note that besides the u and d quarks being more massive than physical, the strange is not finely tuned to its physical value here. Thus, one should not expect these quark's contributions to the polarization scalar and $a_\mu^{\mathrm{HVP,LO}}$ to take on their (precise) physical values. This is all the more true that we leave out the finite renormalization, Z_V, of the local electromagnetic quark current, which contributes only an overall factor to these quantities.

We begin by studying the light, up-down quark contribution to $a_\mu^{\mathrm{HVP,LO}}$. In the left panel of Fig. 1 we show $\Pi_{ud}(Q^2) \equiv \Pi_u(Q^2) = \Pi_d(Q^2)$ vs Q^2 for the four different volumes, obtained using the *usual* and *2nd derivative* (Eq. (2.5)) methods. Also shown are the fits to Padés described above. While for the smallest volume the three methods yield results which differ significantly at low Q^2, this difference reduces as the volume is increased, and the three methods give fully compatible results in the largest volume. This convergence of the methods in the limit of large volumes is also clearly visible in the right panel of Fig. 1, where the values of $a_{\mu,ud}^{\mathrm{HVP,LO}}(Q^2 \le 1\,\mathrm{GeV}^2)$, obtained by integrating the fit functions for $\hat{\Pi}_{ud}(Q^2)$ according to Eq. (2.3), are plotted against $1/M_\pi L$. We choose $M_\pi L$ because it is dimensionless and because $1/M_\pi$ is the longest correlation length in the system. Since the dependence of M_π on L is very weak, the $1/M_\pi L$ dependence shown here is equivalent to a $1/L$ dependence.

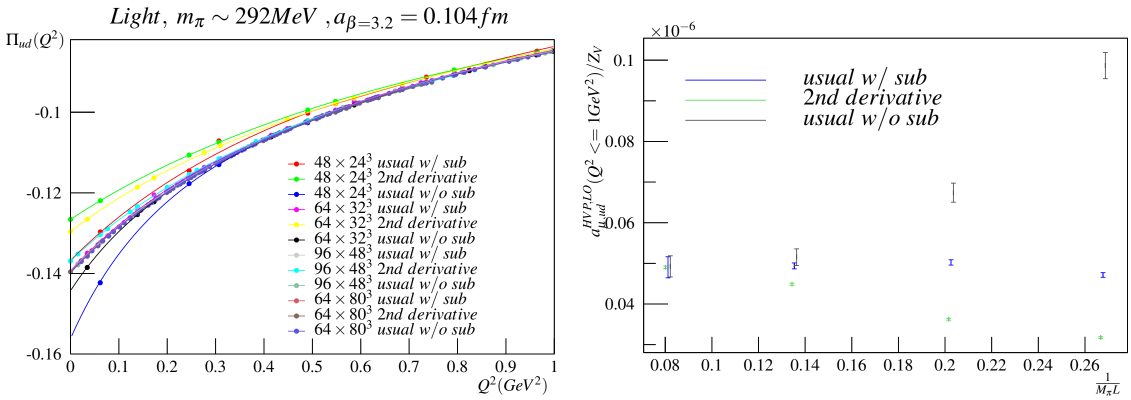

Figure 1: *(Left panel)* $\Pi_{ud}(Q^2)$ vs Q^2 for $M_\pi \sim 292\,\mathrm{MeV}$, as obtained using the *usual* and *2nd derivative* (Eq. (2.5)) methods in four volumes and with all other lattice parameters fixed. The data points are the values obtained from the current-current correlation function and its Fourier derivatives. The curves are the corresponding fits. *(Right panel)* $a_{\mu,ud}^{\mathrm{HVP,LO}}(Q^2 \leq 1\,\mathrm{GeV}^2)$ vs $1/M_\pi L$ obtained from the polarization functions in the left panel.

While results from the three methods converge in the large-volume limit, in smaller volumes the finite-size corrections are significant in some cases. In the smallest volume, with $L = 2.5\,\mathrm{fm}$ or $LM_\pi = 3.7$, the finite-volume correction on $a_{\mu,ud}^{\mathrm{HVP,LO}}(Q^2 \leq 1\,\mathrm{GeV}^2)$, obtained using the *2nd derivative* method, is $\sim 35\%$. It is even larger for the *usual method without subtraction*: around 200%. In the *2nd derivative* case, it is reduced to below 10% by the time $L \gtrsim 5\,\mathrm{fm}$. Only results obtained from the *usual method with subtraction* do the finite-volume effects remain small for all volumes considered.

An interesting feature of the *2nd derivative* method is that it features significantly smaller statistical errors on $a_\mu^{\mathrm{HVP,LO}}(Q^2 \leq 1\,\mathrm{GeV}^2)$ than the *usual method without subtraction*. This remain true to a much smaller extent for the *usual method with subtraction*. In the former case, it is mainly due to the fact that the *2nd derivative* method eliminates the noisy $\Pi_{\mu\nu}^{ud}(0)$, as does *usual method with subtraction*. The additional statistical improvement compared to the *usual method with subtraction* results from the fact that the *2nd derivative* method allows the extraction of $\Pi(Q^2 = 0)$. This constrains the statistical fluctuations of the fitted $\Pi(Q^2)$ vs Q^2 in the very important low-Q^2 region. And though we do not investigate this issue here, this additional constraint will also reduce systematic errors by replacing the usual extrapolation by an interpolation.

We now turn to the strange-quark contribution to $a_\mu^{\mathrm{HVP,LO}}$ and perform the same study of finite-volume effects as for the light contribution. The corresponding results for $\Pi_s(Q^2)$ vs Q^2 and $a_{\mu,s}^{\mathrm{HVP,LO}}(Q^2 \leq 1\,\mathrm{GeV}^2)$ vs $1/M_\pi L$ are shown in Fig. 2. For both quantities, the same general features, as were observed for the light contribution, are seen here. In particular, the results obtained from the *usual method with subtraction* show no volume dependence for the lattices considered. On the other hand, significant finite-volume effects are still observed for the two other methods in smaller volumes, but these disappear as one goes to larger lattices. They are, nevertheless, much smaller than in the light case. For the strange contribution, the finite-volume correction, in the smallest volume with $L = 2.5\,\mathrm{fm}$ or $LM_\pi = 3.7$, is now $\sim 25\%$ on $a_\mu^{\mathrm{HVP,LO}}(Q^2 \leq 1\,\mathrm{GeV}^2)$ obtained using the *derivative* method and $\sim 20\%$ when it is obtained using the *usual approach*

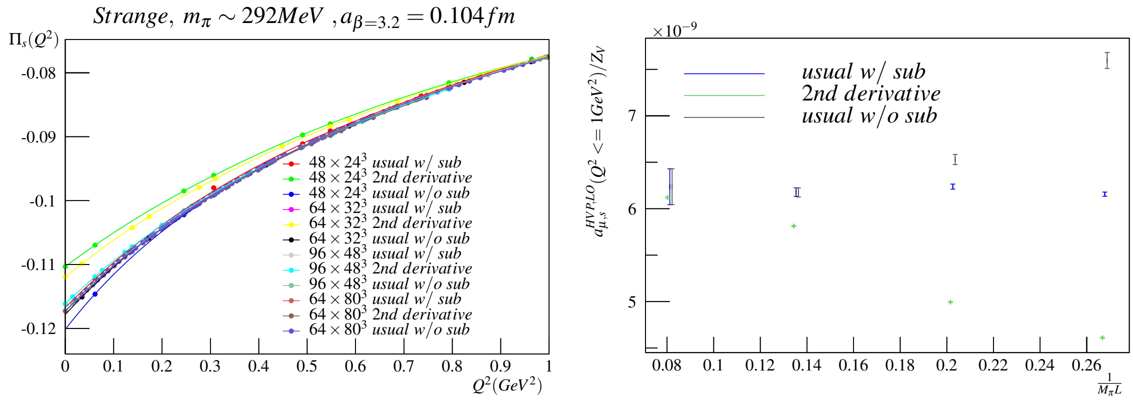

Figure 2: Same as Fig. 1, but for the strange-quark contribution.

without subtraction. By the time one reaches $L = 5.0\,\text{fm}$ or $LM_\pi = 7.4$, the effect is not statistically significant for the *usual* method and below 5% for the *derivative* approach.

4. Conclusion

In addition to the *usual* methods for obtaining the polarization scalar, $\Pi(Q^2)$, which consist in dividing the polarization tensor $\Pi_{\mu\nu}(Q^2)$, or its subtracted counterpart $[\Pi_{\mu\nu}(Q^2) - \Pi_{\mu\nu}(0)]$, by $(Q_\mu Q_\nu - \delta_{\mu\nu}Q^2)$, we have considered a *2nd derivative* method based on Fourier derivatives of quark-electromagnetic-current two-point functions. This method yields $\Pi(Q^2)$ directly, obviating the need to divide by $(Q_\mu Q_\nu - \delta_{\mu\nu}Q^2)$, which clearly is not possible when $Q_\mu = 0$. One advantage of this method is that it gives direct access to $\Pi(0)$. In addition, it does so in a way which is consistent with the results obtained at other values of Q^2. Indeed, the study presented above shows that it may be dangerous to try to combine $\Pi(0)$ obtained through derivatives with $\Pi(Q^2)$, $Q^2 > 0$, obtained in the *usual* ways, as the methods have significantly different systematic errors. The study also shows the advantage of being able to determine $\Pi(0)$ directly: it reduces the statistical errors on the result for the muon anomalous magnetic moment, and a similar reduction is anticipated for the systematic errors.

Using the *usual-without-subtraction*, *usual-with-subtraction* and *2nd derivative* methods for determining $\Pi(Q^2)$, we have conducted a dedicated study of finite-volume effects. We find that the size of finite-volume corrections depends strongly on the method used to obtain $\Pi(Q^2)$, on the quark-contribution considered and, of course, on volume.

The u and d quarks in this study were chosen to be light enough for the ρ to be a resonance in infinite volume. Thus we expect that the physics which governs finite-volume effects here is, at least qualitatively, similar to that which is at play for light quarks at their physical mass. If that is the case, for the contributions of the u, d and s quarks, the *usual method with subtraction* is clearly preferable from the point of view of finite volume effects. As the volume is made larger, the fact that the *2nd derivative* method leads to smaller statistical and $\Pi(Q^2)$-vs-Q^2 fit uncertainties makes it increasingly attractive. Of course, further study is required to understand the extent to which these conclusions carry over to the situation of physical, light-quark masses.

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